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TWO-DIMENSIONAL CONVECTIVE DIFFUSION OF A RADIOACTIVE IMPURITY WITH ALLOWANCE FOR SORPTION IN A POROUS MEDIUM

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A mathematical model of the two-dimensional convective diffusion of radionuclide transfer in a porous medium has been considered. A fundamental solution in the form of Green's function has been obtained for this model. The analytical expressions of the impurity-concentration distribution for stationary conditions and for a few kinds of boundary conditions have been given.

Introduction. Study of the processes and parameters of mass transfer in water-bearing strata is based on the regularities of groundwater-migration theory offering a physicomathematical description of different mechanisms of complex processes of hydrodynamic and physicochemical character.

The migration of chemical components in water-bearing strata occurs within the framework of convective-diffusion processes (with allowance for the mechanism of gravitational differentiation) on which the physicochemical transformations in groundwater and the interaction with enclosing rocks are imposed.

The analytical solutions for longitudinal convective diffusion have been obtained in [1]. Two-dimensional and three-dimensional problems involving the parameters of transversal diffusion arise in actual porous media. Particularly intense scattering of the impurity in a one-dimensional filtration flow is noted on arrival of the contaminant at individual portions of the water-bearing stratum, not at the entire cross section. The process of transversal mixing of liquids has been studied to a lesser extent than longitudinal diffusion. According to [2], one uses the linear dependence on the flow velocity for transversal dispersion; the parameter of transversal scattering is much smaller than the analogous parameter in longitudinal diffusion. Dispersion in different directions gives rise to intricately shaped contamination areals in water-bearing horizons. This is particularly true of the conditions of filtration of contaminated waste liquids from accumulators, tailings and slurry storages, and other types of industrial pools.

Formulation of the Problem. The problem of prediction of the formation of radionuclide-contamination areals with allowance for two-dimensional dispersion can be considered on the basis of a convective-diffusion model [1], in accordance with which the concentration of the radioactive impurity is described by the equation

$$nR\frac{\partial C}{\partial t} = nD_{\mathrm{L}}\frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} + nD_{\mathrm{tr}}\frac{\partial^2 C}{\partial y^2} - u\frac{\partial C}{\partial y} - \lambda nRC\,. \tag{1}$$

To solve it we introduce into consideration the similarity number of mass transfer in migration of radionuclides in a porous medium according to [3]: $z = xv/nD_{\mathrm{L}}$, $w = yu/nD_{\mathrm{tr}}$, $\tau = tv^2/Rn^2D_{\mathrm{L}}$, $\beta = \lambda Rn^2D_{\mathrm{L}}/v^2$, $\mu = u^2D_{\mathrm{L}}/v^2D_{\mathrm{tr}}$, and $S = C/C_0$. In dimensionless coordinates, (1) is reduced to the form

$$\frac{\partial S}{\partial \tau} = \frac{\partial^2 S}{\partial z^2} - \frac{\partial S}{\partial z} + \mu\left(\frac{\partial^2 S}{\partial w^2} - \frac{\partial S}{\partial w}\right) - \beta S\,. \tag{2}$$

We introduce into consideration a new concentration:

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$$a = \exp\left(-\frac{z+w}{2}\right)S. \tag{3}$$

Then, with account for (3), we write

$$\frac{\partial a}{\partial \tau} = \frac{\partial^2 a}{\partial z^2} + \mu \frac{\partial^2 a}{\partial w^2} - \gamma a, \tag{4}$$

where $\gamma = \frac{1}{4} + \frac{1}{4}\mu + \beta$.

Using the substitution $a = b \exp(-\gamma\tau)$ we transform (4) to the ordinary diffusion equation

$$\frac{\partial^2 b}{\partial z^2} + \mu \frac{\partial^2 b}{\partial w^2} = \frac{\partial b}{\partial \tau}, \tag{5}$$

which has the fundamental solution

$$b = \frac{A}{\tau} \exp\left(-\frac{z^2 + \frac{w^2}{\mu}}{4\tau}\right). \tag{6}$$

The constant A in (6) is determined by assigning the mass of the impurity released over the period $d\tau$ in the source:

$$q d\tau = \int_0^H dH \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} b dw. \tag{7}$$

Substituting (6) into (7), we find

$$A = \frac{q d\tau}{4\pi H \sqrt{D_L^* D_{tr}^*}}, \tag{8}$$

where $D_L^* = D_L/R$ and $D_{tr}^* = D_{tr}/R$.

With allowance for what has been stated above, we write the fundamental solution for concentration

$$S = \frac{q dt}{2\pi H \sigma_x \sigma_y} \exp(-\lambda t) \exp\left[-\frac{(x - v^* t)^2}{2\sigma_x^2} - \frac{(y - u^* t)^2}{2\sigma_y^2}\right], \tag{9}$$

where $\sigma_x = \sqrt{2D_L^* t}$, $\sigma_y = \sqrt{2D_{tr}^* t}$, $v^* = \frac{v}{nR}$, and $u^* = \frac{u}{nR}$.

Discussion of Results. The solution (9) determines the line of constant concentration in the form of an ellipse

$$\frac{(x - v^* t)^2}{2\sigma_x^2 d^2} + \frac{(x - u^* t)^2}{2\sigma_y^2 d^2} = 1 \tag{10}$$

with the center at the point $(v^* t,\ u^* t)$, the large axis $2\sqrt{2\sigma_x^2 d^2}$, the small axis $2\sqrt{2\sigma_y^2 d^2}$, the eccentricity $\varepsilon = \sqrt{1 - \sigma_x^2/\sigma_y^2}$, and the focal distance $2\sqrt{2}\, d\sqrt{\sigma_x^2 - \sigma_y^2}$. The foci have coordinates $(d, 0)$, $(-d, 0)$, and

$$d^2 = \ln\left[\frac{q dt}{2\pi H \sqrt{\sigma_x \sigma_y}} \exp(-\lambda t)\right]. \tag{11}$$

Constant-concentration ellipses evolve with time. The reason is that the coordinates of the focus d are exponentially dependent on the decay constant of a radionuclide λ. Furthermore, the contamination areal is substantially dependent on the distribution coefficient K_d which determines dispersion in longitudinal and transverse directions.

The stationary solution of Eq. (2) for $\partial/\partial\tau = 0$ is found using the fundamental solution (6) in which we must replace τ by $\rho = \tau - \tau^*$ (τ^* is the instant of time at which the impurity source acts; thereafter we must integrate the result from 0 to ∞ (continuous arrival of the impurity with a flow rate q at the instant τ) and let it tend to infinity. As a result we obtain

$$a = \frac{q}{4\pi H\sqrt{D_L^* D_{tr}^*}} \int_0^\infty \exp\left(-\frac{N^2}{4\rho} - \gamma\rho\right)\frac{\partial\rho}{\rho}, \quad (12)$$

where $N^2 = z^2 + \frac{w^2}{\mu}$. Transforming thereafter (12) using the substitution $\rho = \frac{N}{2\sqrt{\gamma}}\exp(\xi)$, we find the stationary solution of Eq. (2)

$$S = \frac{q}{2\pi H\sqrt{D_L^* D_{tr}^*}} \exp\left(-\frac{z+w}{2}\right) K_0(N\sqrt{\gamma}), \quad (13)$$

$$K_0(N\sqrt{\gamma}) = \int_0^\infty \exp(-N\sqrt{\gamma}\cosh\xi)\,\partial\xi. \quad (14)$$

Here $K_0(N\sqrt{\gamma})$ is a modified Bessel function of the second kind and of zero order (Macdonald function). In the particular case of a stable impurity $\lambda = 0$ and isotropic dispersion $D_L = D_{tr} = D$ at $R = 1$ expression (14) yields

$$S = \frac{q}{2\pi HD}\exp\left(-\frac{vx + uy}{2D}\right) K_0\left(\frac{rM}{2D}\right), \quad (15)$$

where $r = \sqrt{x^2 + y^2}$ and $M = \sqrt{v^2 + w^2}$ is the filtration-rate modulus. The solution (15) has been obtained by G. I. Marchuk [4].

Green's function (9) enables us to construct the solution of problems for more intricate impurity sources, too. Since Eq. (1) is linear, the superposition principle [3] holds; according to this principle, the propagation of the impurity from individual sources is independent and the total concentration is the sum of the concentrations from each source

$$S = \frac{A}{\tau}\exp(-\lambda t)\int_{-\infty}^{\infty}\exp(-X^2)\int_{-\infty}^{\infty}\exp(-Y^2)\,\varphi(x_0 y_0)\,dx_0 dy_0. \quad (16)$$

Here $X = \frac{x - v^* t - x_0}{\sqrt{2}\,\sigma_x}$, $Y = \frac{y - u^* t - y_0}{\sqrt{2}\,\sigma_y}$, x_0 and y_0 are the integration variables, and $\varphi(x_0, y_0)$ is the impurity distribution at the initial instant of time.

For the radioactive indicator in a transversely homogeneous porous stratum ($\partial/\partial y = 0$) with a stepwise initial distribution, we have $S(X, 0) = 0$ for $|X| > L$, $S(X, 0) = C_0$ for $|X| < L$, and $C \to 0$ for $X \to \infty$. Integrating (16), we obtain

$$S = \frac{1}{2}\exp(-\lambda t)\left[\operatorname{erf}\left(\frac{x - v^* t + L}{\sqrt{2}\sigma_x}\right) - \operatorname{erf}\left(\frac{x - v^* t - L}{\sqrt{2}\sigma_x}\right)\right]. \quad (17)$$

Here erf (ψ) is the probability integral (Laplace function). Equation (17) at $R = 1$ takes the form obtained in [5].

For an extended instantaneous source Q with length L and width W, we can obtain the analytical solution by integrating expression (16) for $C_0 = Q/(LW)$. As a result of the integration we have

$$S = \frac{1}{4}\exp(-\lambda t)\left[\operatorname{erf}\left(\frac{x - v^* t + \frac{L}{2}}{\sqrt{2}\sigma_x}\right) - \operatorname{erf}\left(\frac{x - v^* t - \frac{L}{2}}{\sqrt{2}\sigma_x}\right)\right]\left[\operatorname{erf}\left(\frac{y - u^* t + \frac{W}{2}}{\sqrt{2}\sigma_y}\right) - \operatorname{erf}\left(\frac{y - u^* t - \frac{W}{2}}{\sqrt{2}\sigma_y}\right)\right]. \quad (18)$$

In expression (18), it is assumed that, at the initial instant of time, the coordinates of the source's ends are equal to $(-L/2, L/2)$ and $(-W/2, W/2)$. It is noteworthy that the solution in the form (18) for $u = 0$ is used in the GW SCREEN program [9] for prediction of the contamination of a water-bearing horizon from radioactive-waste storages.

Conclusions. The obtained fundamental solution of the problem of two-dimensional convective diffusion enables one to find analytical solutions for arbitrarily shaped impurity sources.

NOTATION

C, specific activity of a radionuclide in the liquid phase, Bq/liter; C_0, initial specific activity of a radionuclide in the liquid phase, Bq/liter; D, dispersion coefficient, m^2/g; D_L, longitudinal-dispersion coefficient, m^2/g; D_{tr}, transverse-dispersion coefficient, m^2/g; H, thickness of a water-bearing stratum, m; K_d, distribution coefficient of a water-soluble compound, cm^3/kg; L, length of the impurity source, m; n, active porosity of the rock skeleton, m^3/m^3; q, flow rate of the impurity source, m^3/g; Q, activity of the impurity source, Bq; $R = (1 + \rho K_d/n)$, retrogression coefficient; S, dimensionless specific activity; t, time, year; u and v, flow velocity in transverse and longitudinal directions, m/year; W, length of the impurity source, m; w, dimensionless transverse coordinate; x and y, longitudinal and transverse coordinates, m; z, dimensionless longitudinal coordinate; β, dimensionless decay constant of a radionuclide; λ, decay constant of a radionuclide, 1/g; ξ, parameter of the function; ρ, density of the rock skeleton, kg/cm^3; σ_x, dispersion of the impurity in the x direction, m; σ_y, dispersion of the impurity in the y direction, m; τ, dimensionless time. Subscripts: d, distribution; L, longitudinal; 0, initial; tr, transverse.

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